ZERO LOCI OF ADMISSIBLE NORMAL FUNCTIONS WITH TORSION SINGULARITIES

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ABSTRACT. We show that the zero locus of a normal function on a smooth complex algebraic variety S is algebraic provided that the normal function extends to a admissible normal function on a smooth compactification of S with torsion singularity. This result generalizes our previous result for admissible normal functions on curves [arxiv:0604345]. It has also been obtained by M. Saito using a different method in a recent preprint [arXiv:0803.2771v2].

1. INTRODUCTION

Let H be a pure Hodge structure of weight -1 with integral structure $H_{\mathbb{Z}}$. Then, the intermediate Jacobian of H is the complex torus $J(H) = H_{\mathbb{C}}/(F^0 + H_{\mathbb{Z}})$ where F^* is the Hodge filtration of H. If $\mathscr{H}$ is a variation of pure Hodge structure over a complex manifold S with integral structure $\mathscr{H}_{\mathbb{Z}}$, the above construction produces a holomorphic bundle of complex tori $J(\mathscr{H}) \to S$ with fiber $J(\mathscr{H})_s = J(\mathscr{H}_s)$ over s. A normal function ν is a holomorphic section of $J(\mathscr{H})$ which satisfies a version of Griffiths horizontality. Therefore, as a holomorphic section of $J(\mathscr{H})$, the locus of points $\mathscr{Z}$ where ν vanishes is a complex analytic subvariety of S. Furthermore, we have the following conjecture of Griffiths and Green:

Conjecture 1.1. *Let ν be an admissible normal function* [20] *on a smooth complex algebraic variety S. Then, the zero locus $\mathscr{Z}$ of ν is an algebraic subvariety of S.*

In analogy with the work of Cattani, Deligne and Kaplan [4] on the algebraicity of the locus of a Hodge class, an unconditional proof of this conjecture provides evidence in support of the standard conjectures on the existence of filtrations on Chow groups [10]. In the case where S is a curve, we gave an unconditional proof of (1.1) in [3]. Other special cases in which (1.1) is known are normal functions arising from cycles which are algebraically equivalent to zero and the case where the fibers of $J(\mathscr{H})$ are Abelian varieties. In this paper, we prove the following extension of [3]:

Theorem 1.2. *Let ν be an admissible normal function* [20] *on a smooth complex algebraic variety S. Assume that S has a smooth compactification $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor. Then, the zero locus $\mathscr{Z}$ of ν is an algebraic subvariety of S.*

The first step in the proof of Theorem (1.2) is to replace ν be an admissible variation of mixed Hodge structure $\mathscr{V}$ with integral structure $\mathscr{H}_{\mathbb{Z}}$ and weight graded quotients Gr^W_0 and $Gr^W_{-1} = \mathscr{H}$. This is possible by [20]. By a standard construction of Deligne, the mixed Hodge structure on the fiber $\mathscr{V}_s$ defines a

grading $Y(s)$ of the weight filtration of $\mathscr{V}_s$ which preserves the Hodge filtration. The zero locus $\mathscr{Z}$ is then exactly the set of points where $Y(s)$ is defined over $\mathbb{Z}$.

In analogy with [3], the two key technical ingredients in the proof of Theorem (1.2) is the local normal form of a variation of mixed Hodge structure along a normal crossing divisor [19] and the following lemma, which follows from the the full strength of the 1-variable SL_2-orbit theorem [18].

Lemma 1.3. *Let Δ^r be a polydisk and $D \subset \Delta^r$ be a smooth analytic hypersurface. Let $\mathscr{V}$ be a variation of mixed Hodge structure over the complement of D with weight graded quotients Gr^W_0 and Gr^W_{-1}. Assume that the monodromy $T = e^N$ of $\mathscr{V}$ about D is unipotent. Then, for each point $p \in D$, the limit*

$$\hat{Y}(p) = \lim_{s \to p} Y(s)$$

exists, is contained in the kernel of $\mathrm{ad}\,\mathrm{N}$ and has an explicit description in terms of N and the $\mathfrak{sl}_2$-splitting of the limit mixed Hodge structure of $\mathscr{V}$ at p.

Remark 1.4. The limit mixed Hodge structure of $\mathscr{V}$ at p depends upon the choice of local coordinates of $\bar{S}$ at p. However, because the limit $\hat{Y}(p)$ belongs to the kernel of $\mathrm{ad}\,\mathrm{N}$, it well defined independent of the choice of local coordinates.

Alternatively, instead of taking the limit of $Y(s)$ as s accumulates to $p \in D$ along a sequence of points in S, one can twist $Y(s)$ by $e^{-\frac{1}{2\pi i}\log(s)N}$ in analogy with the construction of the limit mixed Hodge structure. This gives a corresponding grading $Y(p)$ which belongs to the kernel of $\mathrm{ad}\,\mathrm{N}$ and has an explicit description in terms of the limit mixed Hodge structure of $\mathscr{V}$ at p. This is stated explicitly in Theorem (4.15) of [18].

In terms of the grading $Y(s)$, the normal function ν is constructed as follows: Let $Y_{\mathbb{Z}}$ be an integral grading of some reference fiber of $\mathscr{V}$. Then, $Y_{\mathbb{Z}}$ extends to a multivalued, integral grading of the weight filtration of $\mathscr{V}$ over S. Therefore, the difference $Y(s) - Y_{\mathbb{Z}}$ is a well defined map from $\mathbb{Z}(0)$ into $J(\mathscr{H}_s)$ for each point $s \in S$. The normal function ν is the image of $1 \in \mathbb{Z}(0)$ under this map. This suggests setting

$$J(\mathscr{H})_p = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), K)$$

where K is the induced mixed Hodge structure on $\ker(N : H_{\mathbb{C}} \to H_{\mathbb{C}})$ and defining

$$\nu(p) = (Y(p) - Y_{\mathbb{Z}})(1) \in \bar{J}(\mathscr{H})_p$$

where $T = e^N$ is the local monodromy of $\mathscr{V}$ at p (assumed unipotent), F_∞ is the limit Hodge filtration of $\mathscr{H}$ at p, and $Y_{\mathbb{Z}}$ is an integral grading of the weight filtration which is invariant under T.

In general, the existence of such a grading $Y_{\mathbb{Z}}$ is obstructed by the class $\sigma_{\mathbb{Z},p}(\nu)$ of ν in the finite group

$$G = \frac{H_{\mathbb{Z}} \cap (T-1)(H_{\mathbb{Q}})}{(T-1)(H_{\mathbb{Z}})} \tag{1.5}$$

In analogy with [12], this allows one to construct a "Neron model" which graphs admissible normal functions on a neighborhood of p: In general, the fibers of $\bar{J}(\mathscr{H})$ can not patch together to form a complex analytic space, since the dimension of $\bar{J}(\mathscr{H})_p$ can be less than the dimension of $J(\mathscr{H})_s$ for $s \in S$. Nonetheless, $\bar{J}(\mathscr{H})$ does carry a kind of generalized complex analytic structure ("slit analytic

space") which traces back to the fundamental work of Kato and Usui compactification of period domains [17]. For recent work in this direction see [21] which uses the Neron model of [12] to give a proof of Theorem 1.2 independent of ours.

Our original interest in the construction of the limits of normal functions is rooted in the work of Griffiths and Green [11] on singularities of normal functions and the Hodge conjecture. Very briefly, the idea of [11] is to start with a smooth projective variety X of complex dimension $2n$ and a very ample line bundle L on X. Let $|L| = \mathbb{P}H^0(X,L)$ and S be the complement of the dual variety $\hat{X} \subset |L|$ of X. Then (cf. [1]), a primitive Deligne cohomology class $\zeta \in H^{2n}_{\mathscr{D}}(X,\mathbb{Z}(n))$ determines an admissible normal function ν on S with cohomology class $\mathrm{cl}_{\mathbb{Z}}(\nu) \in H^1(S,\mathscr{H}_{\mathbb{Z}})$. We then say that ν is singular on $|L|$ if there is a point $p \in \hat{X}$ such that

$$\sigma_{\mathbb{Z},p}(\nu) = \operatorname*{colim}_{p\in U} \mathrm{cl}_{\mathbb{Z}}(\nu)|_{U\cap S} \in \operatorname*{colim}_{p\in U} H^1(S\cap U,\mathscr{H}_{\mathbb{Z}}) \tag{1.6}$$

is non-torsion, where the colimit is taken over all complex analytic neighborhoods U of p in $|L|$. The Hodge conjecture is then equivalent to the following statement [11, 1]

Conjecture 1.7. *For each primitive, non-torsion Hodge class $\zeta \in H^{n,n}(X,\mathbb{Z})$ there exists a positive integer k such that ν is singular on $|L^k|$.*

Remark 1.8. The definition of $\sigma_{\mathbb{Z},p}(\nu)$ is valid for any admissible normal function defined on the complement of a divisor $D \subset \overline{S}$. The finite group (1.5) is exactly the torsion part of the cohomology group appearing in (1.6). In case where D is a smooth divisor, admissibility forces $\sigma_{\mathbb{Z},p}(\nu)$ to be torsion.

Simple examples show that, in general, unless $\sigma_{\mathbb{Z},p}(\nu) = 0$ the limit of $Y(s)$ along a holomorphic arc γ through p depends upon the multiplicities (assumed finite) of the intersection of γ with the irreducible components of the (normal crossing) boundary divisor at p. However, we will show that if $\sigma_{\mathbb{Z},p} = 0$, the limit $Y(s)$ is independent of γ. Furthermore, modulo one step which we shall defer to [2], we obtain the following result:

Theorem 1.9. *Let ν be an admissible normal function on a smooth complex algebraic variety $S \subset \overline{S}$. Assume that $D = \overline{S} - S$ is a normal crossing divisor and that $\sigma_{\mathbb{Z},p}(\nu)$ is torsion for every point $p \in D$. Then, the zero locus of ν is an algebraic subvariety of S.*

Remark 1.10. In fact, the assumption that D is a normal crossing divisor is not necessary. To see this, suppose that we know the result in the case that D is a normal crossing divisor. Let ν be an admissible normal function on $\overline{S}$ which is smooth over S. By Hironaka, we can find a resolution $p : \overline{T} \to \overline{S}$ such that $p^{-1}S \to S$ is an isomorphism and $p^{-1}(\overline{S} \setminus S)$ is a normal crossing divisor. It is easy to see that, if the singularity of ν is zero at every point in $\overline{S}$, then the singularity of the pullback of ν to $\overline{T}$ is zero on $\overline{T}$ as well. Thus, by the theorem, the zero locus of ν on $S = \overline{T} \setminus p^{-1}S$ is algebraic.

Remark 1.11. As mentioned above, Morihiko Saito has obtained an independent proof of Theorem 1.2. He also obtains Theorem 1.9. See [21].

2. Preliminary Results

2.1. **Gradings and Splittings.** Let V be a finite dimensional vector space over a field k of characteristic zero, and

$$0 = L_a \subseteq \cdots \subseteq L_i \subset L_{i+1} \subseteq \cdots \subseteq L_b = V$$

be an increasing filtration of V indexed by $\mathbb{Z}$. Then, a grading of L is a semisimple endomorphism Y of V such that

$$L_i = E_i(Y) \oplus L_{i-1}$$

for each index i, where $E_i(Y)$ is the i-eigenspace of Y. Elements of $\mathrm{GL}(V)$ which preserve L act on gradings of L by the adjoint action:

$$g.Y = gYg^{-1}$$

Let (F,W) be a mixed Hodge structure with Hodge filtration F and weight filtration W. Then [8] there exists a unique, functorial bigrading

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

of the underlying vector space $V_{\mathbb{C}}$ such that

(a) $F^p = \oplus_{r \geq p} I^{r,s}$;
(b) $W_k = \oplus_{r+s \leq k} I^{r,s}$;
(c) $\bar{I}^{p,q} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$.

The associated Deligne grading $Y_{(F,W)}$ of W is the semisimple endomorphism of $V_{\mathbb{C}}$ which acts as multiplication by $p+q$ on $I^{p,q}$. In particular, by properties (a)–(c), if g is an element of $\mathrm{GL}(V_{\mathbb{R}})$ which preserves W then

$$I^{p,q}_{(g.F,W)} = g.I^{p,q}_{(F,W)}$$

with respect to the linear action of $\mathrm{GL}(V)$ on filtrations and subspaces. Likewise, for g as above $Y_{(g.F,W)} = g.Y_{(F,W)}$.

The mixed Hodge structure (F,W) induces a mixed Hodge structure on the Lie algebra $\mathfrak{gl}(V_{\mathbb{C}})$ with associated bigrading

$$\mathfrak{gl}(V_{\mathbb{C}}) = \bigoplus_{p,q} \mathfrak{gl}(V)^{p,q} \tag{2.1}$$

Let λ be an element of the subalgebra

$$\Lambda^{-1,-1} = \bigoplus_{a,b<0} \mathfrak{gl}(V)^{a,b}$$

Then, by properties (a)–(c),

$$I^{p,q}_{(e^{\lambda}.F,W)} = e^{\lambda}.I^{p,q}_{(F,W)} \tag{2.2}$$

and hence $Y_{(e^{\lambda}.F,W)} = e^{\lambda}.Y_{(F,W)}$.

Definition 2.3. A mixed Hodge structure (F,W) is split over $\mathbb{R}$ if $\bar{I}^{p,q} = I^{q,p}$.

Lemma 2.4. *Let (F,W) be a mixed Hodge structure. Then, the following are equivalent:*

(a) *(F,W) is split over $\mathbb{R}$;*
(b) $I^{p,q}_{(F,W)} = F^p \cap \bar{F}^q \cap W_{p+q}$*;*

(c) *There exists a grading Y of W which preserves F and is defined over $\mathbb{R}$, in which case $Y = Y_{(F,W)}$.*

If (F,W) is not split over $\mathbb{R}$ we can construct an associated split mixed Hodge structure $(e^{-i\delta}.F,W)$ as follows:

Theorem 2.5 (Prop (2.20) [6])**.** *There exists a unique real element δ in $\Lambda^{-1,-1}$ such that $\overline{Y}_{(F,W)} = e^{-2i\delta}.Y_{(F,W)}$. Moreover, δ commutes with all (r,r)-morphism of (F,W) and*

$$(e^{-i\delta}.F,W) \tag{2.6}$$

is split over $\mathbb{R}$.

Let W be an increasing filtration of V and N be a nilpotent endomorphism of V which preserves W. Assume that the relative weight filtration [22] M of N and W exists, and suppose that there exists a grading Y_M of M which preserves W and satisfies the condition

$$[Y_M,N] = -2N \tag{2.7}$$

Let Y be a grading of W which preserves M, and

$$N = N_0 + N_{-1} + \cdots$$

be the decomposition of N with respect to $\operatorname{ad} Y$ (i.e. $[Y,N_{-j}] = -jN_{-j}$).

Lemma 2.8. *(Deligne [7, 13]) Under the hypothesis of the previous paragraph, there exists a unique, functorial grading $Y = Y(N,Y_M)$ of W which commutes with Y_M such that:*

(a) *(N_0,H) is an sl_2-pair where $H = Y_M - Y$;*
(b) *If (N_0,H,N_0^+) is the associated sl_2-triple then $[N - N_0, N_0^+] = 0$.*

Corollary 2.9. *For $k > 0$, N_{-k} is either zero or a highest weight vector of weight $k-2$ with respect the representation of sl_2 constructed in the previous lemma. In particular, $N_{-1} = 0$.*

2.2. **Admissible nilpotent orbits.** Let $\mathscr{V} \to S$ be a variation of mixed Hodge structure over a complex manifold. Then [19, 23], in analogy with a variation of pure Hodge structure, a choice of reference fiber V for $\mathscr{V}$ allows us to represent $\mathscr{V}$ by a period map

$$\varphi : S \to \Gamma\backslash\mathscr{M}$$

where $\mathscr{M}$ is a suitable classifying space of graded-polarized mixed Hodge structures and Γ is the image of the monodromy representation. As in the pure case, the classifying space $\mathscr{M}$ is a submanifold of a suitable flag variety, and the period map φ is holomorphic, horizontal and locally liftable. If $F : \tilde{S} \to \mathscr{M}$ is a lifting of φ to the universal cover of S, then

$$\frac{\partial F^p}{\partial z_j} \subseteq F^{p-1}, \qquad \frac{\partial F^p}{\partial \bar{z}_j} \subseteq F^p$$

where $(z_1,\ldots,z_r)$ are local holomorphic coordinates on $\tilde{S}$.

More precisely, let Q_* be the graded-polarizations of Gr^W and $\mathrm{GL}(V)^W$ denote the subgroup of $\mathrm{GL}(V)$ consisting of elements which preserve W. Define

$$G = \{\, g \in \mathrm{GL}(V)^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{R}}(Q_*)\,\}$$

to be the subgroup of $\mathrm{GL}(V)^W$ consisting of elements which act by real isometries of Q on Gr^W. Then, in analogy with the pure case, G acts transitively on $\mathscr{M}$ by biholomorphisms. Likewise, we have an embedding of $\mathscr{M}$ into its "compact dual"

$$\mathscr{M} = G/G^F \hookrightarrow G_{\mathbb{C}}/G^F_{\mathbb{C}} = \check{\mathscr{M}}$$

where $G_{\mathbb{C}} = \{\, g \in \mathrm{GL}(V)^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{C}}(Q_*)\,\}$, and G^F, $G^F_{\mathbb{C}}$ are the corresponding isotopy groups of some point $F \in \mathscr{M}$. The set of points $F \in \mathscr{M}$ for which the corresponding mixed Hodge structure (F,W) is split over $\mathbb{R}$ is a homogeneous space for the Lie group $G_{\mathbb{R}} = G \cap \mathrm{GL}(V_{\mathbb{R}})$. Define $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ to be the respective Lie algebras of $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$.

Let $\Delta \subset \mathbb{C}$ be the unit disk and $\mathscr{V}$ be a variation of mixed Hodge structure on the complement Δ^* of the origin with unipotent monodromy $T = e^N$. Then, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathscr{M} \\ {\scriptstyle s=e^{2\pi i z}}\big\downarrow & & \big\downarrow \\ \Delta^* & \xrightarrow{\varphi} & \Gamma\backslash\mathscr{M} \end{array}$$

Therefore, $\psi(z) = e^{-zN}.F(z) : \Delta^* \to \check{\mathscr{M}}$ descends to a well defined holomorphic map $\psi : \Delta^*$ into $\check{\mathscr{M}}$. If $\mathscr{V}$ is admissible then [22]

(a) $F_\infty = \lim_{s\to 0} \psi(s) \in \check{\mathscr{M}}$ exists;
(b) The relative weight filtration M of N and W exists.

In this case [22],

(i) (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism;
(ii) $(e^{zN}.F_\infty, W)$ is an admissible nilpotent orbit.

For variations of mixed Hodge structure over a higher dimensional base, Kashiwara defined admissibility via a curve test [14]. In particular, if $\mathscr{V}$ is an admissible variation of mixed Hodge structure defined on the complement of a normal crossing divisor with unipotent monodromy transformations $T_j = e^{N_j}$, then the relative weight filtration $M(N_j, W)$ of W and N_j exists for each j.

The remainder of this section is devoted to the discussion of the 1-variable SL_2-orbit theorem [18] which allows us to approximate the nilpotent orbit $\theta(z)$ by an associated SL_2-orbit $\hat{\theta}(z)$ arising from a representation $\rho : \mathrm{SL}_2(\mathbb{R}) \to G_{\mathbb{R}}$. We start by returning to Lemma (2.8):

Lemma 2.10. *(Deligne* [7, 13]*) Let (F,N,W) define an admissible nilpotent orbit with relative weight filtration M. Let $Y_M = Y_{(F,M)}$ and $Y = Y(N, Y_M)$ be the associated grading of Lemma* (2.8)*. Then, Y preserves F. If (F,M) is split over $\mathbb{R}$ then $\bar{Y} = Y$.*

Proof. This follows from the functoriality of Deligne's grading together with an explicit computation in the case where (F,M) is split over $\mathbb{R}$. □

Definition 2.11. A mixed Hodge structure (F,W) is of type (I) if there exists an index i such that $Gr^W_k = 0$ unless $k = i,\, i+1$.

Lemma 2.12. *Every mixed Hodge structure of type (I) is split over $\mathbb{R}$.*

Proof. This follows directly from the short length of the weight filtration and property (c) of Deligne's bigrading. □

Combining the above result, we now obtain a formula for $Y_{(e^{zN}.F,W)}$ along an admissible nilpotent orbit of type (I), i.e. $(e^{zN}.F,W)$ is mixed Hodge structure of type (I) for $Im(z) \gg 0$, when the associated limit mixed Hodge structure (F,M) is split over $\mathbb{R}$:

Theorem 2.13. *Let $(e^{zN}.F,W)$ be an admissible nilpotent orbit of type (I). Let $Y = Y(N,Y_M)$ be the associated grading of W of Lemma* (2.10)*, and suppose that (F,M) is split over $\mathbb{R}$. Then, for $Im(z) > 0$:*

$$Y = Y_{(e^{zN}.F,W)} \tag{2.14}$$

Proof. The fact that $(e^{zN}.F,W)$ is a mixed Hodge structure for $Im(z) > 0$ follows from the fact that (F,M) is split over $\mathbb{R}$ and the results of [6]. By Corollary (2.9) and the short length of W, $N_0 = N$. Therefore, Y preserves $e^{zN}.F$ since $[Y,N] = 0$ and Y preserves F by Lemma (2.10). As Y is defined over $\mathbb{R}$, (2.14) now follows from part (c) of Lemma (2.4). □

The next result allows us to compute the asymptotic behavior of $Y_{(e^{zN}.F,W)}$ along an arbitrary admissible nilpotent orbit of type (I).

Theorem 2.15. *(SL_2-orbit theorem* [18]*) Let $(e^{zN}.F,W)$ be an admissible nilpotent orbit of type (I) with relative weight filtration M. Let $(\tilde{F},M) = (e^{-i\delta}.F,M)$ denote Deligne's δ-splitting* (2.6) *of (F,M). Then, there exists an element*

$$\zeta \in \mathfrak{g}_{\mathbb{R}} \cap \ker(\mathrm{ad}\,\mathrm{N}) \cap \Lambda^{-1,-1}_{(\tilde{\mathrm{F}},\mathrm{M})}$$

and a distinguished real analytic function $\tilde{g} : (a,\infty) \to G_{\mathbb{R}}$ such that

(a) *$e^{iyN}.F = \tilde{g}(y)e^{iyN}.\tilde{F}$ for $y > a$;*

(b) *$\tilde{g}(y)$ and $\tilde{g}^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$\begin{aligned} \tilde{g}(y) &= e^{\zeta}(1 + \tilde{g}_1 y^{-1} + \tilde{g}_2 y^{-2} + \cdots) \\ \tilde{g}^{-1}(y) &= e^{-\zeta}(1 + \tilde{f}_1 y^{-1} + \tilde{f}_2 y^{-2} + \cdots) \end{aligned}$$

with $\tilde{g}_k,\ \tilde{f}_k \in \ker(adN)^{k+1}$;

(c) *δ, ζ and the coefficients $\tilde{g}_k$ are related by the formula*

$$e^{i\delta} = e^{\zeta}\left(1 + \sum_{k>0} \frac{(-i)^k}{k!}(\mathrm{ad}\,\mathrm{N})^{\mathrm{k}}\tilde{\mathrm{g}}_{\mathrm{k}}\right)$$

Let (N_0,H,N_0^+) be the sl_2-triple determined by the sl_2-pair of Lemma (2.10) *and the nilpotent orbit $e^{zN}.\tilde{F}$. The constant ζ can be expressed as a universal Lie polynomial in the Hodge components $\delta^{r,s}$ of δ with respect to $(\tilde{F},M)$. Likewise the coefficients $\tilde{g}_k$ and $\tilde{f}_k$ can be expressed as universal Lie polynomials in the Hodge components $\delta^{r,s}$ and $\mathrm{ad}\,\mathrm{N}_0^+$.*

Remark 2.16. As noted in the proof of Corollary (2.13), for orbits of type (I), $N = N_0$.

For the purpose of computing the asymptotic behavior of the limit grading in §3, it is useful to renormalize the SL_2-orbit theorem as follows: Let

$$g(y) = \tilde{g}(y)e^{-\zeta}, \qquad \hat{F} = e^{\zeta}.\tilde{F}$$

Then, $e^{iyN}.F = g(y)e^{iyN}.\hat{F}$ since $[N,\zeta] = 0$. The mixed Hodge structure $(\hat{F},M)$ is split over $\mathbb{R}$ since $(\tilde{F},M)$ is split over $\mathbb{R}$ and $\zeta \in \mathfrak{g}_{\mathbb{R}}$. Moreover,

$$e^{-\xi} = e^{\zeta}e^{-i\delta} \in \exp(\Lambda^{-1,-1}_{(\hat{F}_\infty,M)})$$

commutes with N and is a universal polynomial in the Hodge components of δ. Likewise, the coefficients

$$g_k = \mathrm{Ad}\,(\mathrm{e}^{\zeta})\tilde{\mathrm{g}}_{\mathrm{k}}$$

of the series expansion

$$g(y) = 1 + \sum_{k>0} g_k y^{-k}$$

are universal polynomials in the Hodge components of δ and $\mathrm{ad}\,\mathrm{N}_0^+$, and satisfy the identity $g_k \in \ker(\mathrm{ad}\,\mathrm{N}_0)^{\mathrm{k}+1}$.

Definition 2.17. Let $(e^{zN}.F,W)$ be a nilpotent orbit of type (I). Then,

$$\hat{F} = e^{-\xi}.F$$

is the sl_2-splitting of (F,M).

Remark 2.18. By virtue of the fact that ζ is given by a universal polynomial in the Hodge components of δ, the sl_2-splitting is defined for any mixed Hodge structure. The formula is as follows [16]: Write the Campbell–Baker–Hausdorff formula as $e^{\alpha}e^{\beta} = e^{H(\alpha,\beta)}$. Then, δ and ξ are related by the formula

$$\delta = H(\xi,-\bar{\xi})/2\sqrt{-1}$$

2.3. **Local normal form.** Let Δ^r be a polydisk with local coordinates $(s_1,\dots,s_r)$ and $\mathscr{V}$ be an admissible variation of mixed Hodge structure on the complement of the divisor $s_1\cdots s_r = 0$ with unipotent monodromy $T_j = e^{N_j}$ about $s_j = 0$. Then, the $I^{p,q}$'s of the limit mixed Hodge structure (F_∞,M) define a vector space complement

$$\mathfrak{q} = \bigoplus_{a<0} \mathfrak{g}^{a,b}$$

to the isotopy algebra $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$, and hence by admissibility, near p we can write the Hodge filtration of $\mathscr{V}$ as

$$F(z) = e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty \tag{2.19}$$

where $\Gamma(s)$ is a $\mathfrak{q}$-valued function which vanishes at $s=0$ and $s_j = e^{2\pi i z_j}$. By horizontality,

$$\frac{\partial}{\partial z_j} F^p(z) \subseteq F^{p-1}(z) \tag{2.20}$$

Let $\wp_a = \oplus_b\, \mathfrak{g}^{a,b}$ and note that:

(i) $\mathfrak{q} = \oplus_{a<0}\, \wp_a$;
(ii) $[\wp_a,\wp_b] \subseteq \wp_{a+b}$;
(iii) $N_j \in \wp_{-1}$.

Inserting (2.19) into (2.20) it then follows that

$$\mathrm{Ad}\,(\mathrm{e}^{-\Gamma(\mathrm{s})})\mathrm{N_j} + 2\pi \mathrm{i} \mathrm{s_j} \mathrm{e}^{-\Gamma(\mathrm{s})} \frac{\partial}{\partial \mathrm{s_j}} \mathrm{e}^{\Gamma(\mathrm{s})} \in \wp_{-1} \tag{2.21}$$

Taking the limit as $s_j \to 0$ in (2.21) it then follows by (i)–(iii) that

$$[\Gamma^{(j)}, N_j] = 0 \tag{2.22}$$

where $\Gamma^{(j)}$ denotes the restriction of $\Gamma(s)$ to the slice $s_j = 0$.

Remark 2.23. In the pure case, this result is due to Cattani and Kaplan [5].

Remark 2.24. The results of this section remains valid in the case where $\mathscr{V}$ is a variation over $\Delta^{*a} \times \Delta^b$ upon setting $N_j = 0$ for $j = a+1, \dots, b$.

2.4. **Intersection Cohomology.** Let $\mathscr{A}_{\mathbb{Q}}$ be a local system of $\mathbb{Q}$-vector spaces over a product of punctured disks Δ^{*r} with unipotent monodromy. Let $A_{\mathbb{Q}}$ be a reference fiber of $\mathscr{A}_{\mathbb{Q}}$ and $N_j \in \mathrm{Hom}(A_{\mathbb{Q}}, A_{\mathbb{Q}})$ denote the monodromy logarithm of $\mathscr{A}_{\mathbb{Q}}$ about the j'th disk. Then, because the N_j's commute, the vector spaces

$$B^p(N_1, \dots, N_r; A_{\mathbb{Q}}) = \bigoplus_{1 \le j_1 < \cdots < j_p \le r} N_{j_1} N_{j_2} \cdots N_{j_p}(A_{\mathbb{Q}}) \tag{2.25}$$

form a complex with respect to the differential d which acts on the summands of (2.25) by the rule

$$d : N_{j_1} \cdots \hat{N}_{j_q} \cdots N_{j_p}(A_{\mathbb{Q}}) \xrightarrow{(-1)^{q-1} N_{j_q}} N_{j_1} \cdots N_{j_p}(A_{\mathbb{Q}}) \tag{2.26}$$

Let $j : \Delta^{*r} \to \Delta^r$ be a holomorphic embedding (the open inclusion) of Δ^{*r} in a product of disks Δ^r and define

$$\mathrm{IH}^p(\Delta^r, \mathscr{A}_{\mathbb{Q}}) = \mathbb{H}^p(\Delta^r, j_{!*}\mathscr{A}_{\mathbb{Q}})$$

Then, by [7, 9] or [15][Corollary 3.4.4]: $H^p(B^*(N_1, \dots, N_r; A_{\mathbb{Q}}) \cong \mathrm{IH}^p(\Delta^r, \mathscr{A}_{\mathbb{Q}})$.

The following result follows from Theorem [1][Lemma 2.1.8]. Here we give a proof that is more in the spirit of the calculations done in this paper.

Theorem 2.27. *Let $\mathscr{V} \to \Delta^{*r}$ be an admissible variation of graded-polarizable mixed Hodge structure with unipotent monodromy which is an extension of $\mathbb{Q}(0)$ by a variation of Hodge structure $\mathscr{H}$ of pure weight -1. Then, the associated short exact sequence*

$$0 \to \mathscr{H}_{\mathbb{Q}} \xrightarrow{\alpha} \mathscr{V}_{\mathbb{Q}} \xrightarrow{\beta} \mathbb{Q}(0) \to 0 \tag{2.28}$$

induces a long exact sequence

$$\cdots \to \mathrm{IH}^{p-1}(\Delta^r, \mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^p(\Delta^r, \mathscr{H}_{\mathbb{Q}}) \xrightarrow{\alpha_*} \mathrm{IH}^p(\Delta^r, \mathscr{V}_{\mathbb{Q}}) \xrightarrow{\beta_*} \mathrm{IH}^p(\Delta^r, \mathbb{Q}(0)) \to \cdots$$

in intersection cohomology.

Proof. Let $\mathscr{V}_{\mathbb{Q}}$ underlie an admissible extension of $\mathbb{Q}(0)$ by a variation of pure Hodge structure $\mathscr{H}$ of weight -1 and $B^*(H_{\mathbb{Q}})$, $B^*(V_{\mathbb{Q}})$, $B^*(\mathbb{Q}(0))$ denote the associated complexes (2.25) defined by the local monodromy. By abuse of notation, let $\mathrm{IH}^p(H_{\mathbb{Q}})$, etc. denote the cohomology of the the corresponding complex. In particular, since each N_j acts trivially on $\mathbb{Q}(0)$,

$$\mathrm{IH}^0(\mathbb{Q}(0)) = \mathbb{Q}(0), \qquad \mathrm{IH}^p(\mathbb{Q}(0)) = 0, \quad p > 0$$

Furthermore, since N_j acts trivially on $\mathbb{Q}(0)$ and $Gr^W_0(V_{\mathbb{Q}}) \cong \mathbb{Q}(0)$ it then follows that

$$N_j(V_{\mathbb{Q}}) \subset W_{-1}(V_{\mathbb{Q}}) \cong H_{\mathbb{Q}}$$

By the existence of the relative weight filtration $M_j = M(N_j, W)$ and the short length of W it then follows[22] that

$$N_j(V_{\mathbb{Q}}) = N_j(H_{\mathbb{Q}})$$

and hence $B^p(V_{\mathbb{Q}}) = B^p(H_{\mathbb{Q}})$ for $p > 0$. Consequently,

$$\mathrm{IH}^p(V_{\mathbb{Q}}) = \mathrm{IH}^p(H_{\mathbb{Q}}), \qquad p > 1$$

Combining the above results, we therefore obtain the exactness of

$$\cdots \to \mathrm{IH}^{p-1}(\mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^p(\mathscr{H}_{\mathbb{Q}}) \xrightarrow{\alpha_*} \mathrm{IH}^p(\mathscr{V}_{\mathbb{Q}}) \xrightarrow{\beta_*} \mathrm{IH}^p(\mathbb{Q}(0)) \to \cdots$$

for $p > 1$.

Thus, in order to complete the proof, it remains to prove the exactness of the sequence

$$0 \to \mathrm{IH}^0(H_{\mathbb{Q}}) \to \mathrm{IH}^0(V_{\mathbb{Q}}) \to \mathrm{IH}^0(\mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^1(H_{\mathbb{Q}}) \to \mathrm{IH}^1(V_{\mathbb{Q}}) \to 0 \tag{2.29}$$

By definition,

$$\mathrm{IH}^0(H_{\mathbb{Q}}) = \cap_j \ker(N_j|_{H_{\mathbb{Q}}}), \qquad \mathrm{IH}^0(V_{\mathbb{Q}}) = \cap_j \ker(N_j)$$

and hence the map $\mathrm{IH}^0(H_{\mathbb{Q}}) \to \mathrm{IH}^0(V_{\mathbb{Q}})$ is injective.

To see that (2.29) is exact at $\mathrm{IH}^0(V_{\mathbb{Q}})$ observe that since $H_{\mathbb{Q}} = W_{-1}(V_{\mathbb{Q}})$ and $\mathbb{Q}(0) = Gr^W_0(V_{\mathbb{Q}})$, the image of $\mathrm{IH}^0(H_{\mathbb{Q}})$ in $\mathrm{IH}^0(V_{\mathbb{Q}})$ is exactly the kernel of the map

$$Gr^W_0 : \mathrm{IH}^0(V_{\mathbb{Q}}) \to \mathrm{IH}^0(\mathbb{Q}(0)) \tag{2.30}$$

For any class $[v] \in IH^0(\mathbb{Q}(0))$,

$$\partial[v] = (N_1(v), \ldots, N_r(v)) \mod dB^0(H_{\mathbb{Q}}) \tag{2.31}$$

where $v \in V_{\mathbb{Q}}$ is any element which projects onto $[v] \in \mathbb{Q}(0) = Gr^W_0(V_{\mathbb{Q}})$. In particular, $\partial[v] = 0$ if and only if there exists $h \in H_{\mathbb{Q}} = W_{-1}(V_{\mathbb{Q}})$ such that

$$N_j(v) = N_j(h)$$

for all j. In this case, $v_o = v - h$ defines an element of $\mathrm{IH}^0(V_{\mathbb{Q}})$ which projects onto of $[v] \in \mathrm{IH}^0(\mathbb{Q}(0))$ under (2.30). As such, (2.29) is exact at $\mathrm{IH}^0(\mathbb{Q}(0))$.

To see that (2.29) is exact at $IH^1(H_{\mathbb{Q}})$ suppose that

$$(N_1(h_1), \ldots, N_r(h_r)), \qquad h_j \in H_{\mathbb{Q}}$$

represents a class $\eta \in \mathrm{IH}^1(H_{\mathbb{Q}})$ which maps to zero under inclusion in $IH^1(V_{\mathbb{Q}})$. Then, there exists a vector $v \in V_{\mathbb{Q}}$ such that

$$N_j(h_j) = N_j(v)$$

for all j, If $v \in H_{\mathbb{Q}}$ then $\eta = 0$. Otherwise, $[v]$ defines a non-zero class in $\mathrm{IH}^0(\mathbb{Q}(0))$ such that $\eta = \partial[v]$. Finally, to verify the surjectivity of the map

$$\mathrm{IH}^1(H_{\mathbb{Q}}) \to \mathrm{IH}^1(V_{\mathbb{Q}})$$

note that $B^p(H_{\mathbb{Q}}) = B^p(V_{\mathbb{Q}})$ for $p > 0$ and $dB^0(H_{\mathbb{Q}}) \subseteq dB^0(V_{\mathbb{Q}})$. □

Definition 2.32. Let $[1]$ be the class of 1 in $\mathrm{IH}^0(\mathbb{Q}(0))$ and $\partial : \mathrm{IH}^0(\mathbb{Q}(0)) \to \mathrm{IH}^1(\mathscr{H}_{\mathbb{Q}})$ be the connecting homomorphism. Then, $\mathrm{sing}_p(v) = \partial 1$.

Remark 2.33. The results of this section remain valid upon replacing $\mathbb{Q}$ by $\mathbb{R}$.

2.5. **Invariant Grading.** Let ν be an admissible normal function over a product of punctured disks $\Delta^{*r} \subset \Delta^r$ with associated variation of mixed Hodge structure $\mathcal{V}$, reference fiber V and nilpotent orbit $\theta(\mathbf{z}) = e^{\sum_j z_j N_j}.F_\infty$. Let $H_{\mathbb{R}} = Gr^W_{-1} V_{\mathbb{R}}$ and $0 = (0,\dots,0) \in \Delta^r$. Let $\hat{\theta}(\mathbf{z}) = e^{\sum_j z_j N_j}.\hat{F}$ be the split (sl_2 or Deligne's δ) orbit attached to $\theta(\mathbf{z})$. Let $\hat{Y}_M$ denote the corresponding grading of $(\hat{F}, M)$ where M is the relative weight filtration of W and the monodromy cone

$$\mathscr{C} = \{ \sum_j a_j N_j \mid a_j > 0 \}$$

Let $\hat{Y} = Y_{(e^{iN}.\hat{F},W)}$, where $N = \sum_j N_j$ Then, by Lemma (2.10), $\hat{Y}$ is real, preserves $\hat{F}$ and commutes with N

Suppose that $\operatorname{sing}_0(\nu) = 0$ and let e_o be the element of $E_0(\hat{Y})$ which projects to $1 \in \mathbb{R}(0)$. Define $e_j = N_j(e_0)$. Then, by (2.31)

$$(e_1, \dots, e_r) \in B^1(H_{\mathbb{R}})$$

is a representative of $\operatorname{sing}_p(\nu)$. Therefore, since $\operatorname{sing}_p(\nu) = 0$ there is an element $f \in H_{\mathbb{R}} = B^0(H_{\mathbb{R}})$ such that $e_j = N_j(f)$. Furthermore, since $e_j = N_j(e_0)$ and $e_0 \in \hat{F}^0$ we have $e_j \in \hat{F}^{-1}$. Therefore, by strictness of morphisms of MHS, we can assume $f \in \hat{F}^0$. Then,

$$e_0 - f = e^{iN}.(e_0 - f) \in e^{iN}.\hat{F}^0$$

Consequently, $e_0 - f$ belongs to $I^{0,0}_{(e^{iN}.\hat{F},W)}$ since $e_0 - f$ is real. On the other hand, by theorem of Deligne e_0 belongs to $I^{0,0}_{(e^{iN}.\hat{F},W)}$. Since Gr^W_0 has rank 1, it then follows that $f = 0$.

Corollary 2.34. $e_0 \in \ker(N_j)$ *for all* j.

Corollary 2.35. *If* $\operatorname{sing}_0(\nu) = 0$ *then*

$$Y_{(e^{\sum_j z_j N_j}.\hat{F},W)} = Y_{(e^{iN}.\hat{F},W)}$$

for $Im(z_1), \dots, Im(z_r) > 0$.

Proof. Since $e_0 \in \ker(N_j)$ for all j, the grading $\hat{Y} = Y_{(e^{iN}.\hat{F},W)}$ commutes with $N_1, \dots, N_r$. Therefore, $\hat{Y}$ is real and preserves $e^{\sum_j z_j N_j}.\hat{F}$ since $\hat{Y}$ preserves $\hat{F}$, and hence is the Deligne grading of $(e^{\sum_j z_j N_j}.\hat{F}, W)$. $\square$

Let $Y_M = Y_{(F,M)}$. Then, $Y_M = e^{i\delta}.\hat{Y}_M$ and $Y = Y(N, Y_M) = e^{i\delta}.\hat{Y}$. Therefore, since $[\delta, N_j] = 0$ for all j, we have:

Corollary 2.36. $[Y, N_j] = 0$ *for all* j.

Definition 2.37. If $\operatorname{sing}_0(\nu) = 0$ we define Y_∞ to be the grading Y of Corollary (2.36). In particular, since Y_∞ commutes with $N_1, \dots, N_r$, it is independent of the choice of local coordinates used in its construction.

3. Limit Gradings

Let $D \subset \bar{S}$ be a smooth divisor, $p \in D$ and Δ^r be an analytic polydisk in $\bar{S}$ containing p. Pick local coordinates $(s_1, \dots, s_r)$ on Δ^r such that $D \cap \Delta^r$ is given by $s_1 = 0$. Represent ν by an admissible variation of mixed Hodge structure $\mathcal{V}$ over $\Delta^* \times \Delta^{r-1}$ with weight graded quotients $Gr^W_0 = \mathbb{Z}(0)$ and $Gr^W_{-1} = \mathscr{H}$. Assume

that the local monodromy of $\mathscr{V}$ about D is given by a unipotent transformation $T = e^N$. Let

$$F(z; s_2, \dots, s_r) : U \times \Delta^{r-1} \to \mathscr{M}$$

be a lifting of the local period map of $\mathscr{V}$ where U is the upper half-plane.

Let

$$F(z; s_2, \dots, s_r) = e^{zN} e^{\Gamma(s)}.F_\infty$$

be the local normal form of the period map of $\mathscr{V}$ at p. Let $\Gamma_0(s) = \Gamma(0, s_2, \dots, s_r)$ and

$$F_\infty(s_2, \dots, s_r) = e^{\Gamma_0(s)}.F_\infty$$

Let W be the weight filtration of $\mathscr{V}$, M be the relative weight filtration of N and W. Then,

$$\theta(z; s_2, \dots, s_r) = e^{zN}.F_\infty(s_2, \dots, s_r)$$

is an admissible nilpotent orbit in 1-variable which depends complex analytically upon the parameters $(s_2, \dots, s_r) \in \Delta^{r-1}$. Let

$$(\hat{F}_\infty(s_2, \dots, s_r), M) = (e^{-\xi(s_1, \dots, s_r)}.F_\infty(s_2, \dots, s_r), M)$$

denote the sl_2-splitting of $(F_\infty(s_2, \dots, s_r), M)$. Then, ξ is real analytic in $(s_2, \dots, s_r)$ since it is given by universal Lie polynomials in the Hodge components of Deligne's δ-splitting of $(F_\infty(s_2, \dots, s_r), M)$.

By the SL_2-orbit theorem (2.15)

$$\theta(iy; s_2, \dots, s_r) = g(y; s_2, \dots, s_r) e^{iyN}.\hat{F}_\infty(s_2, \dots, s_r)$$

where

$$g(y; s_2, \dots, s_r) = (1 + \sum_{k>0} g_k(s_2, \dots, s_r) y^{-k})$$

belongs to $G_\mathbb{R}$ and the coefficients $g_k(s_2, \dots, s_r)$ are real analytic in $(s_2, \dots, s_r)$ since they are given by universal Lie polynomials.

We now derive an asymptotic formula for $Y_{(F(z; s_2, \dots, s_r), W)}$. Write $z = x + iy$. Then,

$$\begin{aligned} Y_{(F(z; s_2, \dots, s_r), W)} &= Y_{(e^{xN} e^{iyN} e^{\Gamma(s)}.F_\infty, W)} \\ &= e^{xN}.Y_{(e^{iyN} e^{\Gamma(s)} e^{-\Gamma_0(s)} e^{\Gamma_0(s)}.F_\infty, W)} \\ &= e^{xN}.Y_{(e^{iyN} e^{\Gamma(s)} e^{-\Gamma_0(s)}.F_\infty(s_2, \dots, s_r), W)} \end{aligned}$$

Let $e^{\Gamma_1(s)} = e^{\Gamma(s)} e^{-\Gamma_0(s)}$ and note that $s_1 | \Gamma_1$ in $\mathscr{O}(\Delta^r)$. Then,

$$\begin{aligned} Y_{(F(z; s_2, \dots, s_r), W)} &= e^{xN}.Y_{(e^{iyN} e^{\Gamma_1(s)}.F_\infty(s_2, \dots, s_r), W)} \\ &= e^{xN}.Y_{(\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)}).\theta(iy; s_2, \dots, s_r), W)} \\ &= e^{xN}.Y_{(\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)}) g(y; s_2, \dots, s_r) e^{iyN}.\hat{F}_\infty(s_2, \dots, s_r), W)} \end{aligned}$$

Let $F_o(s_2, \dots, s_r) = e^{iN}.\hat{F}_\infty(s_2, \dots, s_r)$ and

$$Y_1 = Y_{(F_o(s_2, \dots, s_r), W)}, \qquad Y_2 = Y_{(\hat{F}_\infty(s_2, \dots, s_r), M)}$$

Then, by Corollary (2.13):

$$Y_{(e^{iyN}.\hat{F}_\infty(s_2, \dots, s_r), W)} = Y_1$$

Likewise, since N and $H = Y_2 - Y_1$ is an sl_2-pair:

$$e^{iyN}.\hat{F}_\infty(s_2, \dots, s_r) = y^{-H/2}.F_o(s_2, \dots, s_r)$$

Note that Y_1 and H depend real analytically on $(s_2,\dots,s_r)$.

Lemma 3.1. *Let* $\gamma(y) = \mathrm{Ad}(e^{-iyN})g(y;s_2,\dots,s_r)$. *Then,* $\lim_{y\to\infty}\gamma(y)$ *exists, and is real analytic in* $(s_2,\dots,s_r)$.

Proof. This follows directly from the fact that $g_k(s_2,\dots,s_r)$ is real analytic in $(s_2,\dots,s_r)$ and $g_k \in \ker(\mathrm{ad}\,N)^{k+1}$. $\square$

Returning to the calculation of $Y_{(F(z;s_2,\dots,s_r),W)}$, and abbreviating $g(y;s_2,\dots,s_r)$ to $g(y)$, we have

$$\begin{aligned} Y_{(F(z;s_2,\dots,s_r),W)} &= e^{xN}.Y_{(\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)})g(y)e^{iyN}.\hat{F}_\infty(s_2,\dots,s_r),W)} \\ &= e^{xN}.Y_{(g(y)e^{iyN}\gamma^{-1}(y)e^{\Gamma_1(s)}\gamma(y).\hat{F}_\infty(s_2,\dots,s_r),W)} \end{aligned}$$

Let $e^{\Gamma_2} = \mathrm{Ad}(\gamma^{-1}(y))e^{\Gamma_1}$ and recall that $s_1|\Gamma_1$. Therefore,

$$\begin{aligned} Y_{(F(z;s_2,\dots,s_r),W)} &= e^{xN}.Y_{(g(y)e^{iyN}e^{\Gamma_2(s)}.\hat{F}_\infty(s_2,\dots,s_r),W)} \\ &= e^{xN}g(y)y^{-H/2}.Y_{(e^{iN}\mathrm{Ad}(y^{H/2})(e^{\Gamma_2}).\hat{F}_\infty(s_2,\dots,s_r),W)} \end{aligned}$$

where $\mathrm{Ad}(y^{H/2})\Gamma_2$ can be uniformly bounded by a constant times $y^c e^{-2\pi y}$ as $y\to\infty$ for some constant c.

We now prove Lemma (1.3) of the introduction, which we shall use in the next section to prove the algebraicity of the zero locus $\mathscr{Z}$. Modulo our discussion of dependence on parameters, this essentially the same calculation use to prove the existence of the limit grading in [3].

Theorem 3.2. *Let* $(s_1(m),\dots,s_r(m))$ *be a sequence of points in* $\Delta^*\times\Delta^{r-1}$ *which converges to* $(0,s_2,\dots,s_r)$ *as* $m\to\infty$. *Let* $(z(m),s_2(m),\dots,s_r(m))$ *be a lifting of this sequence to* $U\times\Delta^{r-1}$ *with the real part of* z *restricted to an interval of finite length. Then,*

$$\lim_{m\to\infty} Y_{(F(z(m);s_2(m),\dots,s_r(m)),W)} = Y_{(e^{iN}.\hat{F}_\infty(s_2,\dots,s_r),W)}.$$

Proof. Suppress the dependence of $(z(m);s_2(m),\dots,s_r(m))$ on m. By the previous results:

$$Y_{(F(z;s_2,\dots,s_r),W)} = e^{xN}g(y)y^{-H/2}.Y_{(e^{iN}\mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)}).\hat{F}_\infty(s_2,\dots,s_r),W)} \tag{3.3}$$

where $\mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)}$ is uniformly bounded by some constant times $y^c e^{-2\pi y}$. Therefore,

$$Y_{e^{iN}\mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)}).\hat{F}_\infty(s_2,\dots,s_r),W)} = Y_{(e^{iN}.\hat{F}_\infty(s_2,\dots,s_r),W)} + \alpha \tag{3.4}$$

where α is uniformly bounded by $y^c e^{-2\pi y}$. The result now follows by inserting (3.4) into (3.3) and taking the limit at $m\to\infty$, since $H = H(s_2,\dots,s_r)$ commutes with $Y_1(s_2,\dots,s_m)$. $\square$

In order to construct the limit normal function we need the following analog of Theorem (3.2) where we twist the grading $Y_{(\mathscr{F},\mathscr{W})}$ by $e^{-\frac{1}{2\pi i}\log(s_1)N}$. Again, modulo dependence on parameters, this is really just a glorified version of Theorem (4.15) in [18].

Theorem 3.5. *Let $(s_1(m),\dots,s_r(m))$ be a sequence of points in $\Delta^*\times\Delta^{r-1}$ which converges to $(0,s_2,\dots,s_r)$ as $m\to\infty$. Let $(z(m),s_2(m),\dots,s_r(m))$ be a lifting of this sequence to $U\times\Delta^{r-1}$ with the real part of z restricted to an interval of finite length. Then,*

$$\lim_{m\to\infty} e^{-zN}.Y_{(F(z(m);s_2(m),\dots,s_r(m)),W)} = Y(N,Y_{(F_\infty(s_2,\dots,s_r),M)})$$

where $Y(N,Y_{(F_\infty(s_2,\dots,s_r),M)})$ is the grading of Lemma (2.10).

Proof. We repeat the argument of the proof of Theorem (3.2) to obtain

$$\begin{aligned} e^{-zN}.Y_{(F(z;s_2,\dots,s_r),W)} &= e^{-iyN}g(y)y^{-H/2}.(Y_{(e^{iN}.\hat{F}_\infty(s_2,\dots,s_r),W)}+\alpha) \\ &= e^{-iyN}\tilde{g}(y)e^{iyN}e^{-\zeta}e^{iyN}y^{-H/2}.(Y_{(e^{iN}.\hat{F}_\infty(s_2,\dots,s_r),W)}+\alpha) \end{aligned} \tag{3.6}$$

By part (c) of the SL_2-orbit theorem (cf. equation (4.19) in [18]), we have

$$\lim_{y\to\infty} \mathrm{Ad}(\mathrm{e}^{-\mathrm{i}y\mathrm{N}})\tilde{\mathrm{g}}(\mathrm{y}) = \mathrm{e}^{\zeta}\left(1+\sum_{\mathrm{k}>0}\frac{(-\mathrm{i})^{\mathrm{k}}}{\mathrm{k}!}(\mathrm{ad}\,\mathrm{N})^{\mathrm{k}}\tilde{\mathrm{g}}_{\mathrm{k}}\right) = \mathrm{e}^{\mathrm{i}\delta}$$

Therefore, as in the proof of Theorem (3.2) it follows that

$$\lim_{m\to\infty} e^{-zN}.Y_{(F(z;s_2,\dots,s_r),W)} = e^{i\delta}e^{-\zeta}.Y_{(e^{iN}.\hat{F}_\infty(s_2,\dots,s_r),W)} \tag{3.7}$$

where δ and ζ are real-analytic in $(s_2,\dots,s_r)$. By (2.10) and (2.13)

$$\begin{aligned} Y_{(e^{iN}.\hat{F}(s_2,\dots,s_r),W)} &= Y(N,Y_{(\hat{F}(s_2,\dots,s_r),M)}) \\ &= Y(N,e^{\zeta}e^{-i\delta}.Y_{(F_\infty(s_2,\dots,s_r),M)}) \\ &= e^{\zeta}e^{-i\delta}.Y(N,Y_{(F_\infty(s_2,\dots,s_r),M)}) \end{aligned} \tag{3.8}$$

by the functoriality of Deligne's construction. Inserting (3.8) into (3.7) completes the proof. □

Remark 3.9. By virtue of the functoriality of the grading $Y(N,Y_M)$ with respect to the pair (N,Y_M) and the fact that $Y(N,Y_M)\in\ker(\mathrm{ad}\,\mathrm{N})$ due to the short length of W, it follows that $Y(N,Y_{(F_\infty(s_2,\dots,s_r),M)})$ is independent of the choice of local coordinates.

In connection with the proof of Theorem (1.9), we now consider the case where ν is an admissible normal function, on $\Delta^{*r}\subseteq\Delta^r$ with unipotent monodromy, and $\mathrm{sing}_0(\nu)=0$. Let $(s_1(m),\dots,s_r(m))$ be a sequence of points in Δ^{*r} which converge to $0=(0,\dots,0)$. Let $(z_1(m),\dots,z_r(m))$ be a lifting of this sequence to the product of upper half-planes, with the real parts of each $z_j(m)$ restricted to an interval of finite length. Then, we want to compute

$$\lim_{m\to\infty} Y_{(F(z_1(m),\dots,z_r(m)),W)}$$

where $F(z_1,\dots,z_r)$ is a lifting of the local period map to U^r. Suppose that (after passage to a subsequence)

$$\lim_{m\to\infty}\frac{y_{j+1}(m)}{y_j(m)}\in(0,\infty) \tag{3.10}$$

for $j=1,\dots,r-1$. Then, exactly the same arguments as above show that

$$\lim_{m\to\infty} Y_{(F(z_1(m),\dots,z_r(m)),W)} = Y(N,Y_{(\hat{F}_\infty,M)})$$

where N is any element in the monodromy cone $\mathscr{C} = \{\sum_j a_j N_j \mid a_j > 0\}$. The key point is that:

(a) By Corollary (2.35), $\hat{Y} = Y(N, Y_{(\hat{F}_\infty, M)})$ is independent of N.

(b) Under the hypothesis of condition (3.10), the element

$$N(y_1,\dots,y_r) = N_1 + \frac{y_2}{y_1}N_2 + \cdots + \frac{y_r}{y_1}N_r$$

remains within a compact subset of $\mathscr{C}$ as $m \to \infty$. Therefore,

$$e^{iy_1 N(y_1,\dots,y_r)}.F_\infty = g(y_1)e^{iy_1 N(y_1,\dots y_r)}.\hat{F}_\infty$$

where all the coefficients of g depend real-analytically on $N(y_1,\dots,y_r)$, since Deligne's construction (2.8) is algebraic in the pair (N, Y_M).

In general, by reordering the variables if necessary, one can always pass to some subsequence such that

$$\lim_{m\to\infty} \frac{y_{j+1}(m)}{y_j(m)} \in [0,\infty) \tag{3.11}$$

for $j = 1,\dots,r-1$. Suppose for simplicity that $\lim_{m\to\infty} \frac{y_{j+1}(m)}{y_j(m)} = 0$. Then, the main theorem of [16] asserts that

$$\lim_{m\to\infty} Y_{(e^{iy_1N_1+\cdots+iy_rN_r}.F_\infty, W)} \tag{3.12}$$

exists (independent of any assumptions about $\mathrm{sing}(\nu) = 0$).

Theorem 3.13. *Assume that* $\mathrm{sing}(\nu) = 0$ *and that* $(z_1(m),\dots,z_r(m))$ *is a sequence of points in* U^r *which satisfies condition* (3.11)*. Then,*

$$\lim_{m\to\infty} Y_{(F(z_1(m),\dots,z_r(m)),W)} = Y(\sum_j N_j, Y_{(\hat{F}_\infty,M)})$$

Proof. This is basically just the main result of [16] together with dependence on parameters (see the proof of the norm estimates in [16]) and Corollary (2.35). The details will appear in [2]. □

4. Algebraicity of the Zero Locus

We now prove Theorem (1.2). Let $\mathscr{Z}$ be the zero locus of an admissible normal function ν on a smooth complex algebraic variety S which admits a smooth compactification $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor. Let $p \in D$ be an accumulation point of $\mathscr{Z}$, and $(s_1,\dots,s_r)$ be local coordinates on a polydisk $\Delta^r \subset \bar{S}$ containing p, relative to which D is given by the equation $s_1 = 0$. Let $\mathscr{V} \to \Delta^* \times \Delta^{r-1}$ be an admissible variation of mixed Hodge structure which represents ν on $S \cap \Delta^r$. Without loss of generality, assume that $\mathscr{V}$ has unipotent monodromy.

Let $(s_1(m),\dots,s_r(m))$ be a sequence of points in $\mathscr{Z}$ which converge to p, and

$$F(z; s_2,\dots,s_r) : U \times \Delta^{r-1} \to \mathscr{M}$$

be a lifting of the period map of $\mathscr{V}$, where U is the upper half-plane. Let $(z(m), s_2(m),\dots,s_r(m))$ be a lifting of $(s_1(m),\dots,s_r(m))$ to $U \times \Delta^{r-1}$ with the real part of z restricted to an interval of finite length. Then, by Theorem (3.2)

$$\lim_{m\to\infty} Y_{(F(z(m);s_2(m),\dots,s_r(m)),W)} = Y_1(0,\dots,0) \tag{4.1}$$

In particular, since the set of integral gradings is discrete, equation (4.1) forces

$$Y_{\mathbb{Z}} = Y_1(0,\dots,0)$$

to be an integral grading of W. By Lemma (2.10) and Corollary (2.13), it then follows that

(a) $Y_{\mathbb{Z}} \in \ker(\operatorname{ad} N)$;
(b) $Y_{\mathbb{Z}}$ preserves the Hodge filtration $\hat{F}_\infty = e^{-\xi}.F_\infty$ of the sl_2-splitting the limit mixed Hodge structure (F_∞, M);
(c) $\xi \in \ker(\operatorname{ad} N) \cap \Lambda^{-1,-1}_{(\hat{F}_\infty,M)}$.

Let $Y_\infty = e^{\xi}.Y_{\mathbb{Z}}$. Then, Y_∞ preserves F_∞ and belongs to $\ker(\operatorname{ad} N)$. Therefore, due to the short length of the weight filtration, there exists a unique $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$-valued function $f(z;s_2,\dots,s_r)$ such that

$$Y_{(F(z;s_2,\dots,s_r),W)} = e^{zN}e^{\Gamma(s)}.(Y_\infty + f)$$

The local defining equation for $\mathscr{Z}$ near p is therefore

$$Y_{\mathbb{Z}} = e^{zN}e^{\Gamma(s)}.(Y_\infty + f) \tag{4.2}$$

Transposing the $e^{zN}e^{\Gamma(s)}$ factor over to the other side, we then obtain,

$$e^{-\Gamma(s)}.Y_{\mathbb{Z}} = Y_\infty + f \tag{4.3}$$

The subalgebra $\mathfrak{q}$ is closed under the action of $\operatorname{ad} Y_\infty$. Consequently,

$$Y_{\mathbb{Z}} = e^{-\xi}.Y_\infty = Y_\infty + \lambda$$

for some element $\lambda \in \mathfrak{q}$. More properly, by equation (2.2), $\Lambda^{-1,-1}_{(\hat{F},M)} = \Lambda^{-1,-1}_{(F,M)}$, wherefrom the result follows since $\Lambda^{-1,-1}_{(F,M)}$ is closed under $\operatorname{ad} Y_\infty$.

Accordingly, (4.3) reduces to

$$e^{-\Gamma(s)}.(Y_\infty + \lambda) = Y_\infty + f \tag{4.4}$$

Again, because Y_∞ grades W and $\operatorname{ad} Y_\infty$ preserves $\mathfrak{q}$, we have

$$e^{-\Gamma(s)}.Y_\infty = Y_\infty + \alpha(s)$$

for some holomorphic function $\alpha(s)$ with values in $\mathfrak{q} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$. Recalling that $W_{-1}\mathfrak{g}_{\mathbb{C}}$ acts simply transitively on the gradings of W, it then follows that equation (4.4) simplifies to

$$e^{-\Gamma(s)}.(Y_\infty + \lambda) = Y_\infty$$

since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{q} \oplus \mathfrak{g}_{\mathbb{C}}^{F_\infty}$ and f takes values in $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$. Clearly, this equation is complex analytic on Δ^r. It also forces $\lambda = 0$.

Granting Theorem (3.13), the proof of Theorem (1.9) is identical: A sequence of points $(s_1,\dots,s_r)$ converging to a point $p \in D$ where $\operatorname{sing}_p(\nu) = 0$ forces

$$Y_{\mathbb{Z}} = Y_{(\hat{F}_\infty,M)}$$

to be an integral grading which in the kernel of $\operatorname{ad} N_j$ for each j. Repeating the argument given above, it then follows that the local defining equation for the zero locus is $e^{-\Gamma(s)}.Y_\infty = Y_\infty$.

Remark 4.5. The above arguments also show that if ν is an admissible normal function on $S = \bar{S} - D$ and p is a smooth point of D such that $\sigma_{\mathbb{Z},p}(\nu)$ is non-zero torsion then p can not be an accumulation point of $\mathscr{Z}$.

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